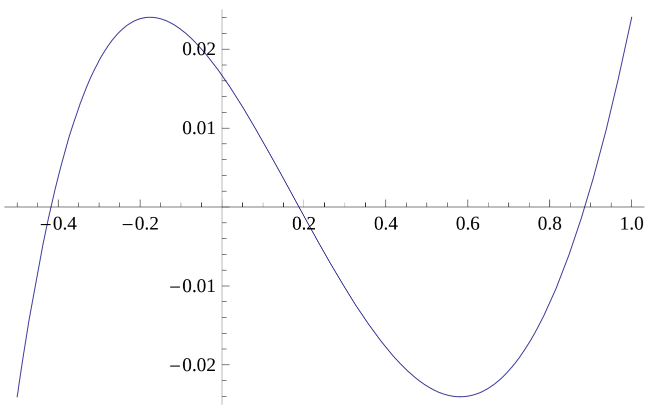
```
In[1]:= (*Reciprocal square root with a quadratic approximation followed by a 2nd-
         order Householder iteration.*)

In[2]:= fi[x_] := pi.NestList[# * x &, 1, Length[pi] - 1]

In[3]:= f[x_] := 1 / Sqrt[(x + 1) / 2]

In[4]:= xi = N[(Table[-Cos[i * π / 3], {i, 0, 3}] * 3 + 1) / 4]

Out[4]= {-0.5, -0.125, 0.625, 1.}

In[17]:= pi =
          {p0, p1, p2} /. (Solve[({p0, p1, p2}.{1, #, #^2} & /@ xi) - (f /@ xi) == {ϵ, -ϵ, ϵ, -ϵ} * (f /@ xi),
              {p0, p1, p2, ϵ}])[[1]]

Out[17]= {1.4378, -0.823394, 0.409642}

         {1.437799046117536‘, -0.823394375837328‘, 0.4096419668459485‘}

In[18]:= ((fi /@ xi) - (f /@ xi)) / (f /@ xi)

Out[18]= {-0.0240466, 0.0240466, -0.0240466, 0.0240466}

In[19]:= xi =
          Join[{xi[[1]]}, # - ((D[(fi[x] - f[x]) / f[x], x] / D[(fi[x] - f[x]) / f[x], x, x] /. x → #) & /@ #) &[
            Take[xi, {2, -2}]], {xi[[-1]]}]

Out[19]= {-0.5, -0.175472, 0.581492, 1.}

In[20]:= Plot[(fi[x] - f[x]) / f[x], {x, -0.5, 1}]
```

Out[20]=



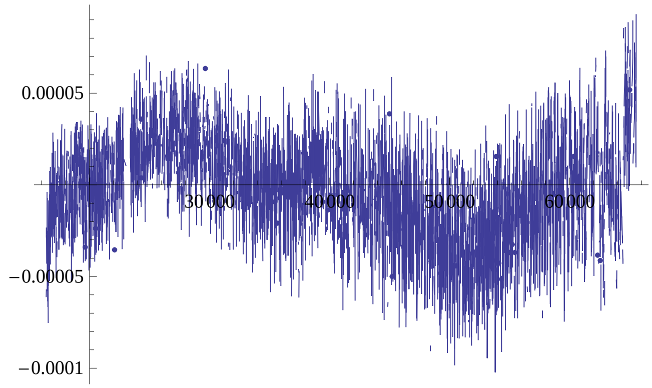

```
In[21]:= Round[pi * 16 384]

Out[21]= {23 557, -13 490, 6712}

In[22]:= mq14[a_, b_] := Floor[a * b * (1. / 2^14)]

In[23]:= mq15[a_, b_] := Floor[a * b * (1. / 2^15)]

In[24]:= rsqrt[x_] := Module[{n, r, y},
           n = x - 32 768;
           r = 23 557 + mq15[n, -13 490 + mq15[n, 6713]];
           y = 2 * (mq15[mq15[r, r], n] + mq15[r, r] - 16 384);
           r + mq15[r, mq15[y, -16 384 + mq15[y, 12 288]]]
          ]

In[25]:= rsqrt /@ {16 384, 16 385, 16 386, 16 387, 16 388, 16 389, 16 390, 16 391}

Out[25]= {32 766, 32 766, 32 765, 32 763, 32 762, 32 761, 32 760, 32 759}
```

In[26]:=
```
rsqrt /@ {65535, 65534, 65533, 65532, 65531, 65530, 65529, 65528}
```

Out[26]=
```
{16385, 16385, 16385, 16386, 16386, 16386, 16386, 16386}
```

In[27]:=
```
Plot[(rsqrt[x] - 16384 / Sqrt[x / 65536]) / (16384 / Sqrt[x / 65536]), {x, 16384, 65535}]
```

Out[27]=



In[28]:=
```
Sqrt[Mean[
  Table[N[(rsqrt[x] - 16384 / Sqrt[x / 65536]) / (16384 / Sqrt[x / 65536])]^2, {x, 16384, 65535}]]]
```

Out[28]=
```
0.0000280979
```

In[29]:=
```
Max[
 Table[Abs[N[(rsqrt[x] - 16384 / Sqrt[x / 65536]) / (16384 / Sqrt[x / 65536])]], {x, 16384, 65535}]]
```

Out[29]=
```
0.000104956
```

In[30]:=
```
Max[Table[Abs[N[rsqrt[x] - 16384 / Sqrt[x / 65536]]], {x, 16384, 65535}]]
```

Out[30]=
```
2.26591
```

In[31]:=
```
(*Reciprocal square root with a linear approximation followed by two Newton
 iterations. This is the same amount of work, but is not as accurate as the above.*)
```

In[32]:=
```
xi = N[(Table[-Cos[i * π / 2], {i, 0, 2}] * 3 + 1) / 4]
```

Out[32]=
```
{-0.5, 0.25, 1.}
```

In[45]:=
```
pi =
 {p0, p1} /. (Solve[({p0, p1}.{1, #} & /@ xi) - (f /@ xi) == {ϵ, -ϵ, ϵ} * (f /@ xi), {p0, p1, ϵ}])[[1]]
```

Out[45]=
```
{1.52341, -0.609364}
```

```
{1.5234089594072642`, -0.6093635837629056`}
```

In[46]:=
```
((fi /@ xi) - (f /@ xi)) / (f /@ xi)
```

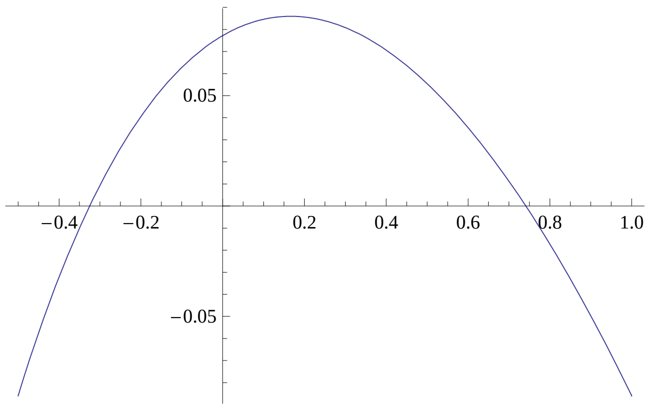
Out[46]=
```
{-0.0859546, 0.0859546, -0.0859546}
```

In[47]:=
```
xi =
 Join[{xi[[1]]}, # - ((D[(fi[x] - f[x]) / f[x], x] / D[(fi[x] - f[x]) / f[x], x, x] /. x → #) & /@ #) &[
   Take[xi, {2, -2}]], {xi[[-1]]}]
```

Out[47]=
```
{-0.5, 0.166667, 1.}
```

In[48]:=
```
Plot[(fi[x] - f[x]) / f[x], {x, -0.5, 1}]
```

Out[48]=



In[49]:=
```
Round[pi * 16 384]
```

Out[49]=
```
{24 960, -9984}
```

In[50]:=
```
rsqrt[x_] := Module[{n, r, y},
  n = x - 32 768;
  r = 24 960 + mq15[n, -9985];
  y = mq15[mq15[r, r], n] + mq15[r, r] - 16 384;
  r = r - mq15[r, y];
  y = mq15[mq15[r, r], n] + mq15[r, r] - 16 384;
  r = r - mq15[r, y]
 ]
```

In[51]:=
```
rsqrt /@ {16 384, 16 385, 16 386, 16 387, 16 388, 16 389, 16 390, 16 391}
```

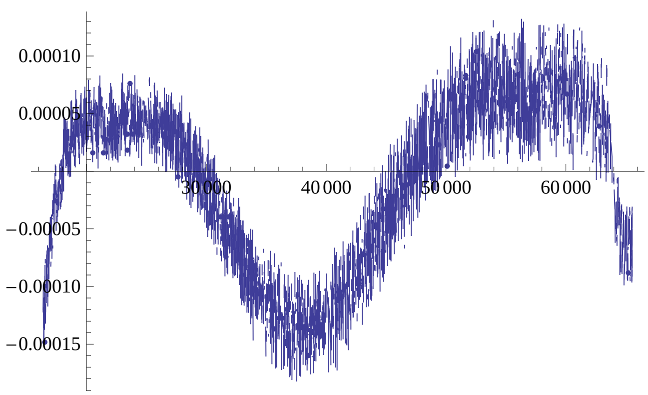
Out[51]=
```
{32 764, 32 763, 32 762, 32 761, 32 760, 32 759, 32 758, 32 757}
```

In[52]:=
```
rsqrt /@ {65 535, 65 534, 65 533, 65 532, 65 531, 65 530, 65 529, 65 528}
```

Out[52]=
```
{16 383, 16 383, 16 383, 16 383, 16 384, 16 384, 16 384, 16 384}
```

In[53]:=
```
Plot[(rsqrt[x] - 16 384 / Sqrt[x / 65 536]) / (16 384 / Sqrt[x / 65 536]), {x, 16 384, 65 535}]
```

Out[53]=



In[54]:=
```
Sqrt[Mean[
  Table[N[(rsqrt[x] - 16 384 / Sqrt[x / 65 536]) / (16 384 / Sqrt[x / 65 536])]^2, {x, 16 384, 65 535}]]]
```

Out[54]=
```
0.0000722421
```

In[55]:=
```
Max[
 Table[Abs[N[(rsqrt[x] - 16 384 / Sqrt[x / 65 536]) / (16 384 / Sqrt[x / 65 536])]], {x, 16 384, 65 535}]]
```

Out[55]=
```
0.000187016
```

In[56]:=
```
Max[Table[Abs[N[rsqrt[x] - 16384 / Sqrt[x / 65536]]], {x, 16384, 65535}]]
```
Out[56]= 5.08058

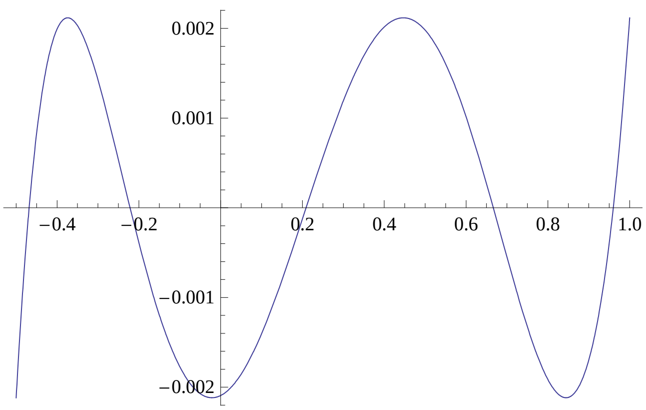
In[57]:=
```
(*Reciprocal square root with pure polynomial approximations. These can be less work,
but are nowhere near as accurate, even when using the same number of mulitplies as above.*)
```

In[58]:=
```
xi = N[(Table[-Cos[i * π / 5], {i, 0, 5}] * 3 + 1) / 4]
```
Out[58]= {-0.5, -0.356763, 0.0182373, 0.481763, 0.856763, 1.}

In[68]:=
```
pi = {p0, p1, p2, p3, p4} /.
  (Solve[({p0, p1, p2, p3, p4}.{1, #, #^2, #^3, #^4} & /@ xi) - (f /@ xi) ==
      {ϵ, -ϵ, ϵ, -ϵ, ϵ, -ϵ} * (f /@ xi), {p0, p1, p2, p3, p4, ϵ}])[[1]]
```
Out[68]= {1.41125, -0.702385, 0.601043, -0.545571, 0.237777}

In[69]:=
```
((fi /@ xi) - (f /@ xi)) / (f /@ xi)
```
Out[69]= {-0.00211772, 0.00211772, -0.00211772, 0.00211772, -0.00211772, 0.00211772}

In[70]:=
```
xi = Join[{xi[[1]]},
  # - ((D[(fi[x] - f[x]) / (f[x]), x] / D[(fi[x] - f[x]) / (f[x]), x, x] /. x → #) & /@ #) &[
   Take[xi, {2, -2}]], {xi[[-1]]}]
```
Out[70]= {-0.5, -0.373965, -0.0214785, 0.446693, 0.844448, 1.}

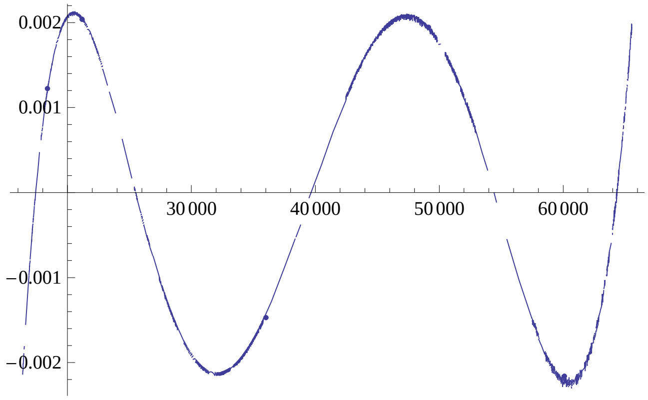
In[71]:=
```
Plot[(fi[x] - f[x]) / f[x], {x, -0.5, 1}]
```
Out[71]=


In[72]:=
```
Round[16384 * pi]
```
Out[72]= {23122, -11508, 9847, -8939, 3896}

In[73]:=
```
rsqrtp[x_] := Module[{n},
  n = x - 32768;
  23122 + mq15[n, -11508 + mq15[n, 9847 + mq15[n, -8939 + mq15[n, 3896]]]]
 ]
```

```
In[74]:= Plot[(rsqrtp[x] - 16384 / Sqrt[x / 65536]) / (16384 / Sqrt[x / 65536]), {x, 16384, 65535}]
```

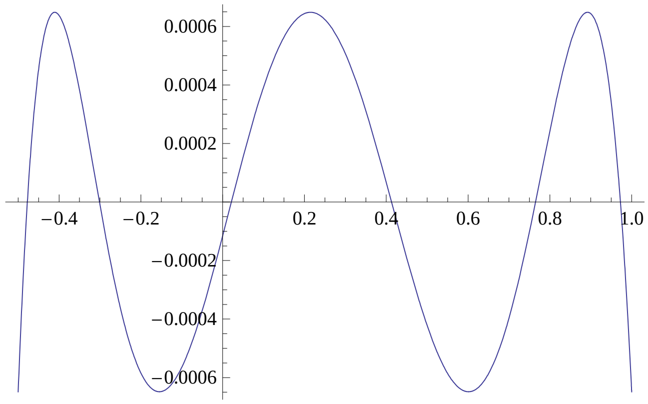
Out[74]=




```
In[75]:= Sqrt[Mean[Table[
            N[(rsqrtp[x] - 16384 / Sqrt[x / 65536]) / (16384 / Sqrt[x / 65536])]^2, {x, 16384, 65535}]]]

Out[75]= 0.00149898

In[76]:= Max[Table[
          Abs[N[(rsqrtp[x] - 16384 / Sqrt[x / 65536]) / (16384 / Sqrt[x / 65536])]], {x, 16384, 65535}]]

Out[76]= 0.00230869

In[77]:= Max[Table[Abs[N[rsqrtp[x] - 16384 / Sqrt[x / 65536]]], {x, 16384, 65535}]]

Out[77]= 70.0002

In[78]:= xi = N[(Table[-Cos[i * π / 6], {i, 0, 6}] * 3 + 1) / 4]

Out[78]= {-0.5, -0.399519, -0.125, 0.25, 0.625, 0.899519, 1.}

In[91]:= pi = {p0, p1, p2, p3, p4, p5} /.
          (Solve[({p0, p1, p2, p3, p4, p5}.{1, #, #^2, #^3, #^4, #^5} & /@ xi) - (f /@ xi) ==
              {ϵ, -ϵ, ϵ, -ϵ, ϵ, -ϵ, ϵ} * (f /@ xi), {p0, p1, p2, p3, p4, p5, ϵ}])[[1]]

Out[91]= {1.41405, -0.699353, 0.530116, -0.53289, 0.478055, -0.190627}

In[92]:= ((fi /@ xi) - (f /@ xi)) / (f /@ xi)

Out[92]= {-0.000648318, 0.000648318, -0.000648318,
          0.000648318, -0.000648318, 0.000648318, -0.000648318}

In[93]:= xi =
          Join[{xi[[1]]}, # - ((D[(fi[x] - f[x]) / f[x], x] / D[(fi[x] - f[x]) / f[x], x, x] /. x → #) & /@ #) &[
            Take[xi, {2, -2}]], {xi[[-1]]}]

Out[93]= {-0.5, -0.410543, -0.154716, 0.214843, 0.601119, 0.892047, 1.}
```

In[94]:= **Plot[(fi[x] - f[x]) / f[x], {x, -0.5, 1}]**

Out[94]=



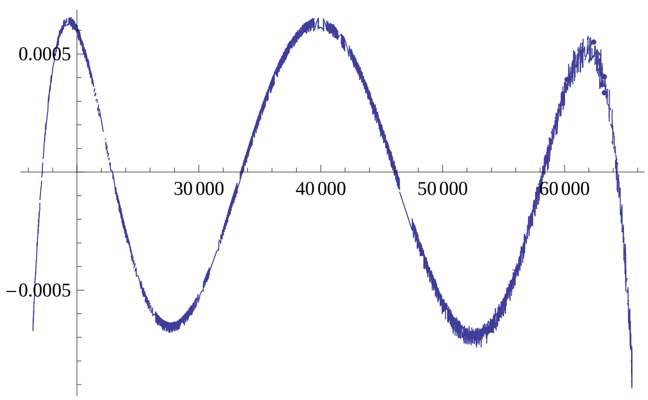
In[95]:= **Round[16 384 * pi]**

Out[95]= {23 168, -11 458, 8685, -8731, 7832, -3123}

In[96]:=
```
rsqrtp[x_] := Module[{n},
  n = x - 32 768;
  23 168 + mq15[n, -11 458 + mq15[n, 8685 + mq15[n, -8731 + mq15[n, 7832 + mq15[n, -3123]]]]]
 ]
```

In[97]:= **Plot[(rsqrtp[x] - 16 384 / Sqrt[x / 65 536]) / (16 384 / Sqrt[x / 65 536]), {x, 16 384, 65 535}]**

Out[97]=



In[98]:=
```
Sqrt[Mean[Table[
   N[(rsqrtp[x] - 16 384 / Sqrt[x / 65 536]) / (16 384 / Sqrt[x / 65 536])]^2, {x, 16 384, 65 535}]]]
```

Out[98]= 0.000455923

In[99]:=
```
Max[Table[
  Abs[N[(rsqrtp[x] - 16 384 / Sqrt[x / 65 536]) / (16 384 / Sqrt[x / 65 536])]], {x, 16 384, 65 535}]]
```

Out[99]= 0.000915477

In[100]:= **Max[Table[Abs[N[rsqrtp[x] - 16 384 / Sqrt[x / 65 536]]], {x, 16 384, 65 535}]]**

Out[100]= 22.0022

In[101]:= **xi = N[(Table[-Cos[i * π / 7], {i, 0, 7}] * 3 + 1) / 4]**

Out[101]= {-0.5, -0.425727, -0.217617, 0.0831093, 0.416891, 0.717617, 0.925727, 1.}

In[111]:=
```
pi = {p0, p1, p2, p3, p4, p5, p6} /.
  (Solve[({p0, p1, p2, p3, p4, p5, p6}.{1, #, #^2, #^3, #^4, #^5, #^6} & /@ xi) - (f /@ xi) ==
     {ϵ, -ϵ, ϵ, -ϵ, ϵ, -ϵ, ϵ, -ϵ} * (f /@ xi), {p0, p1, p2, p3, p4, p5, p6, ϵ}])[[1]]
```

Out[111]= {1.41446, -0.70585, 0.517164, -0.450303, 0.494537, -0.425383, 0.155575}

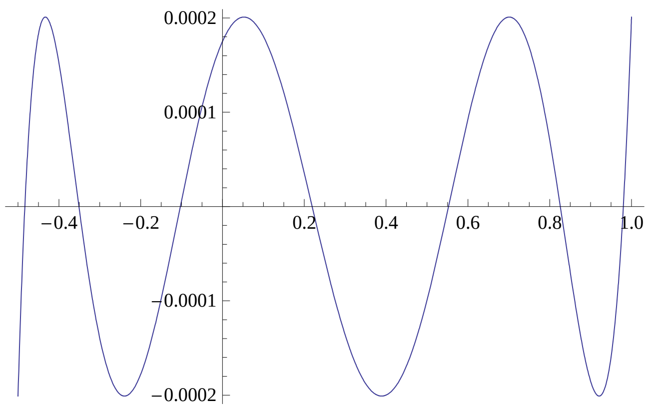
In[112]:=
```
((fi /@ xi) - (f /@ xi)) / (f /@ xi)
```

Out[112]= {-0.000200963, 0.000200963, -0.000200963, 0.000200963, -0.000200963, 0.000200963, -0.000200963, 0.000200963}

In[113]:=
```
xi =
 Join[{xi[[1]]}, # - ((D[(fi[x] - f[x]) / f[x], x] / D[(fi[x] - f[x]) / f[x], x, x] /. x → #) & /@ #) &[
   Take[xi, {2, -2}]], {xi[[-1]]}]
```

Out[113]= {-0.5, -0.433174, -0.239699, 0.0523423, 0.389095, 0.701094, 0.920881, 1.}

In[114]:=
```
Plot[(fi[x] - f[x]) / f[x], {x, -0.5, 1}]
```

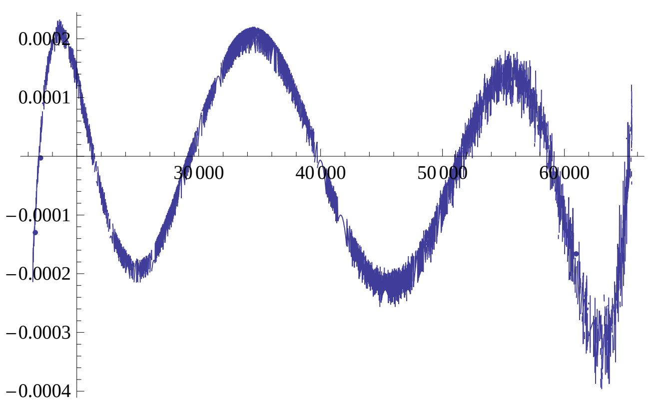
Out[114]=


In[115]:=
```
Round[16 384 * pi]
```

Out[115]= {23 175, -11 565, 8473, -7378, 8103, -6969, 2549}

In[116]:=
```
rsqrtp[x_] := Module[{n},
  n = x - 32 768;
  23 175 + mq15[n,
    -11 565 + mq15[n, 8473 + mq15[n, -7378 + mq15[n, 8103 + mq15[n, -6969 + mq15[n, 2549]]]]]]
 ]
```

In[117]:=
```
Plot[(rsqrtp[x] - 16 384 / Sqrt[x / 65 536]) / (16 384 / Sqrt[x / 65 536]), {x, 16 384, 65 535}]
```

Out[117]=


In[118]:=
```
Sqrt[Mean[Table[
   N[(rsqrtp[x] - 16384 / Sqrt[x / 65536]) / (16384 / Sqrt[x / 65536])]^2, {x, 16384, 65535}]]]
```

Out[118]= 0.000154571

In[119]:=
```
Max[Table[
  Abs[N[(rsqrtp[x] - 16384 / Sqrt[x / 65536]) / (16384 / Sqrt[x / 65536])]], {x, 16384, 65535}]]
```

Out[119]= 0.000428171

In[120]:=
```
Max[Table[Abs[N[rsqrtp[x] - 16384 / Sqrt[x / 65536]]], {x, 16384, 65535}]]
```

Out[120]= 7.18896

In[123]:=
```
(*Polynomial approximation of a square root.*)
```

In[124]:=
```
f[x_] := Sqrt[(x + 1) / 2]
```

In[125]:=
```
xi = N[(Table[-Cos[i * π / 5], {i, 0, 5}] * 3 + 1) / 4]
```

Out[125]= {-0.5, -0.356763, 0.0182373, 0.481763, 0.856763, 1.}

In[144]:=
```
pi = {p0, p1, p2, p3, p4} /.
  (Solve[({p0, p1, p2, p3, p4}.{1, #, #^2, #^3, #^4} & /@ xi) - (f /@ xi) ==
      {ϵ, -ϵ, ϵ, -ϵ, ϵ, -ϵ} * (f /@ xi), {p0, p1, p2, p3, p4, ϵ}])[[1]]
```

Out[144]= {0.70724, 0.352788, -0.0918906, 0.0518601, -0.0202491}

```
{0.7072403335147993`, 0.35278823789299407`,
 -0.09189057085305771`, 0.051860055091630206`, -0.020249053852030158`}
```

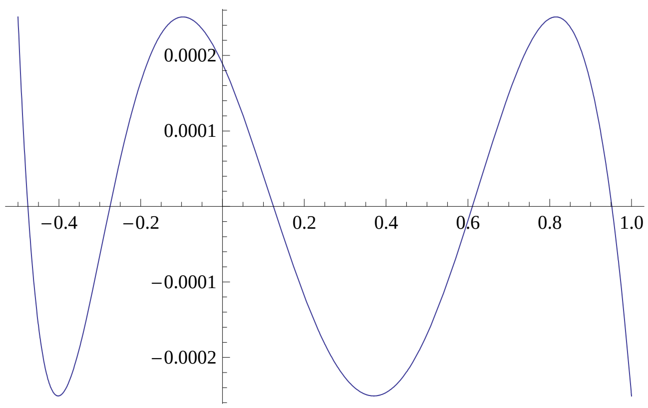
In[145]:=
```
((fi /@ xi) - (f /@ xi)) / (f /@ xi)
```

Out[145]= {0.000250998, -0.000250998, 0.000250998, -0.000250998, 0.000250998, -0.000250998}

In[146]:=
```
xi = Join[{xi[[1]]},
  # - ((D[(fi[x] - f[x]) / (f[x]), x] / D[(fi[x] - f[x]) / (f[x]), x, x] /. x → #) & /@ #) &[
   Take[xi, {2, -2}]], {xi[[-1]]}]
```

Out[146]= {-0.5, -0.401682, -0.0969033, 0.369975, 0.815118, 1.}

In[147]:=
```
Plot[(fi[x] - f[x]) / f[x], {x, -0.5, 1}]
```

Out[147]=


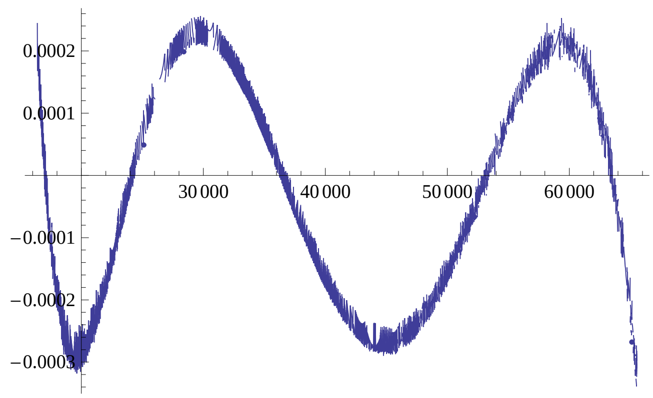
In[148]:=
```
Round[32 768 * pi]
```

Out[148]=
```
{23 175, 11 560, -3011, 1699, -664}
```

In[149]:=
```
sqrtp[x_] := Module[{n},
  n = x - 32 768;
  23 175 + mq15[n, 11 561 + mq15[n, -3011 + mq15[n, 1699 + mq15[n, -664]]]]
 ]
```

In[150]:=
```
Plot[(sqrtp[x] - Sqrt[x / 65 536] * 32 768) / (Sqrt[x / 65 536] * 32 768), {x, 16 384, 65 535}]
```

Out[150]=



In[151]:=
```
Sqrt[Mean[
  Table[N[(sqrtp[x] - Sqrt[x / 65 536] * 32 768) / (Sqrt[x / 65 536] * 32 768)]^2, {x, 16 384, 65 535}]]]
```

Out[151]=
```
0.000175912
```

In[152]:=
```
Max[Table[
  Abs[N[(sqrtp[x] - Sqrt[x / 65 536] * 32 768) / (Sqrt[x / 65 536] * 32 768)]], {x, 16 384, 65 535}]]
```

Out[152]=
```
0.000335749
```

In[153]:=
```
Max[Table[Abs[N[(sqrtp[x] - Sqrt[x / 65 536] * 32 768)]], {x, 16 384, 65 535}]]
```

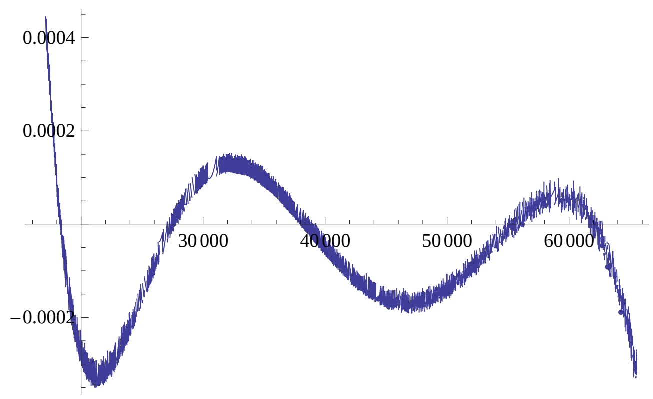
Out[153]=
```
10.999
```

In[154]:=
```
(*JM's original square root approximation for comparison.*)
```

In[155]:=
```
sqrtp[x_] := Module[{n},
  n = x - 32 768;
  23 174 + mq15[n, 11 584 + mq15[n, -3011 + mq15[n, 1570 + mq15[n, -557]]]]
 ]
```

In[156]:=
```
Plot[(sqrtp[x] - Sqrt[x / 65 536] * 32 768) / (Sqrt[x / 65 536] * 32 768), {x, 16 384, 65 535}]
```

Out[156]=



In[157]:=
```
Sqrt[Mean[
  Table[N[(sqrtp[x] - Sqrt[x / 65 536] * 32 768) / (Sqrt[x / 65 536] * 32 768)]^2, {x, 16 384, 65 535}]]]
```

Out[157]=
```
0.000157979
```

In[158]:=
```
Max[Table[
  Abs[N[(sqrtp[x] - Sqrt[x / 65 536] * 32 768) / (Sqrt[x / 65 536] * 32 768)]], {x, 16 384, 65 535}]]
```

Out[158]=
```
0.000884886
```

In[159]:=
```
Max[Table[Abs[N[(sqrtp[x] - Sqrt[x / 65 536] * 32 768)]], {x, 16 384, 65 535}]]
```

Out[159]=
```
14.5002
```

In[160]:=
```
(*Polynomial approximation of cosine.*)
```

In[161]:=
```
f[x_] := Cos[Sqrt[x] * π / 2]
```

In[162]:=
```
xi = N[Table[-Cos[i * π / 4], {i, 0, 4}] + 1] / 2
```

Out[162]=
```
{0., 0.146447, 0.5, 0.853553, 1.}
```

In[172]:=
```
pi = {p0, p1, p2, p3} /.
  (Solve[({p0, p1, p2, p3}.{1, #, #^2, #^3} & /@ xi) - (f /@ xi) == {-ϵ, ϵ, -ϵ, ϵ, -ϵ},
     {p0, p1, p2, p3, ϵ}])[[1]]
```

Out[172]=
```
{0.999993, -1.23348, 0.25258, -0.0190957}
```

```
{0.9999932952821674‘, -1.233484503785825‘, 0.2525802391345619‘, -0.01909573534873664‘}
```

In[173]:=
```
(fi /@ xi) - (f /@ xi)
```

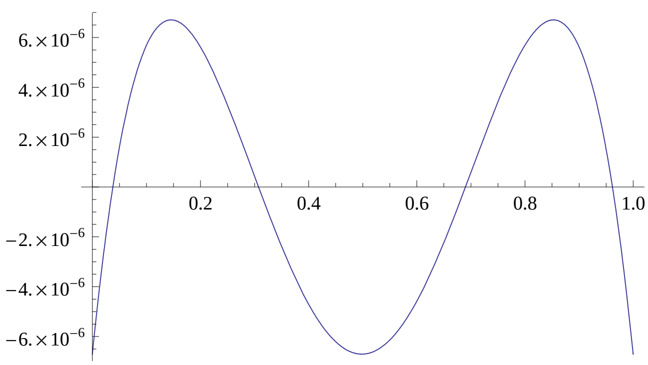
Out[173]= $\{-6.70472\times10^{-6}, 6.70472\times10^{-6}, -6.70472\times10^{-6}, 6.70472\times10^{-6}, -6.70472\times10^{-6}\}$

In[174]:=
```
xi = Join[{xi[[1]]},
  # - ((D[fi[x] - f[x], x] / D[fi[x] - f[x], x, x] /. x → #) & /@ #) &[Take[xi, {2, -2}]], {xi[[-1]]}]
```

Out[174]=
```
{0., 0.145581, 0.498265, 0.852684, 1.}
```

In[175]:=
```
Plot[fi[x] - f[x], {x, 0, 1}]
```

Out[175]=



In[176]:=
```
Round[32 768 * pi] + {0, 32 768, 0, 0}
```

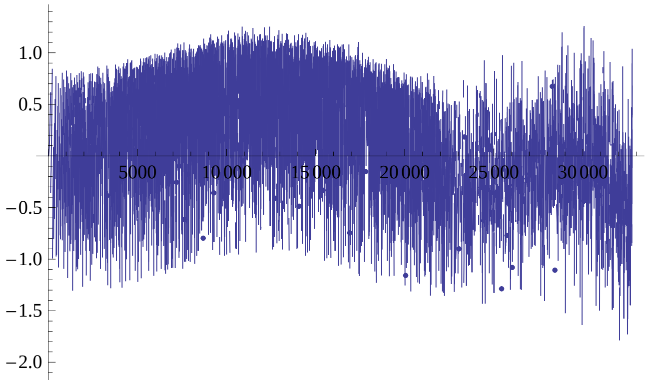
Out[176]=
```
{32 768, -7651, 8277, -626}
```

In[177]:=
```
mq15p[a_, b_] := Round[a * b * (1. / 2^15)]
```

In[178]:=
```
cospi2p[x_] := Module[{x2},
  x2 = mq15p[x, x] - 1;
  32 767 - x2 + mq15[x2, -7650 + mq15[x2, 8277 + mq15[x2, -626]]]
 ]
```

In[179]:=
```
Plot[(cospi2p[x] - Cos[x * π / 65 536] * 32 768), {x, 0, 32 767}]
```

Out[179]=



In[180]:=
```
Sqrt[Mean[Table[N[(cospi2p[x] - Cos[x * π / 65 536] * 32 768)]^2, {x, 0, 32 767}]]]
```

Out[180]= 0.508555

In[181]:=
```
Max[Table[Abs[N[(cospi2p[x] - Cos[x * π / 65 536] * 32 768)]], {x, 0, 32 767}]]
```

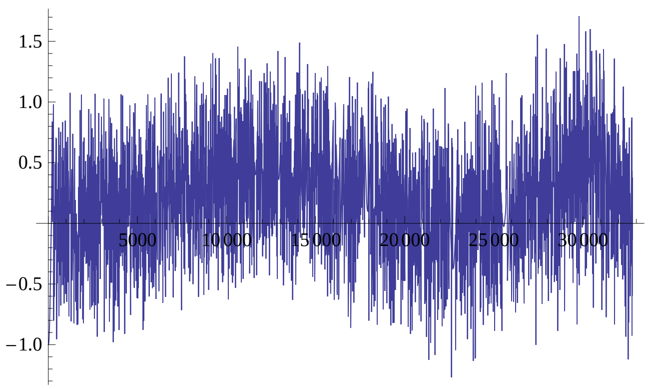
Out[181]= 2.00436

In[182]:=
```
(*This is JM's original approximation,
which was pretty good, if slightly more complicated.*)
```

In[183]:=
```
cospi2p[x_] := Module[{x2},
  x2 = mq15p[x, x];
  1 + Min[32 766, 32 767 - x2 + mq15p[x2, -7651 + mq15p[x2, 8277 + mq15p[x2, -626]]]]
 ]
```

In[184]:=
```
Plot[(cospi2p[x] - Cos[x * π / 65 536] * 32 768), {x, 0, 32 767}]
```

Out[184]=



In[185]:=
```
Sqrt[Mean[Table[N[(cospi2p[x] - Cos[x * π / 65 536] * 32 768)]^2, {x, 0, 32 767}]]]
```

Out[185]= 0.528566

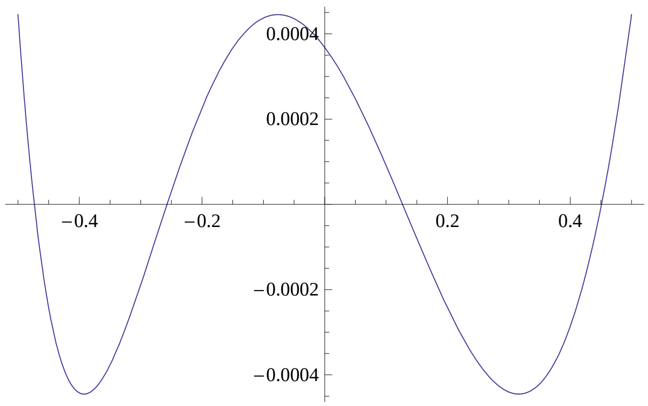
In[186]:=
```
Max[Table[Abs[N[(cospi2p[x] - Cos[x * π / 65 536] * 32 768)]], {x, 0, 32 767}]]
```

Out[186]= 1.91708

In[187]:=
```
(*Polynomial approximation of the binary logarithm.*)
```

In[188]:=
```
f[x_] := Log[2, (2 * x + 3) / 4]
```

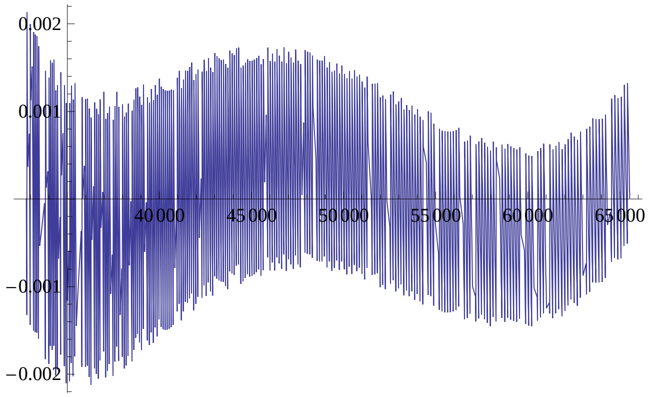
```
In[189]:= xi = N[(Table[-Cos[i*π/4], {i, 0, 4}])/2]

Out[189]= {-0.5, -0.353553, 0., 0.353553, 0.5}

In[205]:= pi = {p0, p1, p2, p3} /.
            (Solve[({p0, p1, p2, p3}.{1, #, #^2, #^3} & /@ xi) - (f /@ xi) == {ϵ, -ϵ, ϵ, -ϵ, ϵ} * (2 + f /@ xi),
               {p0, p1, p2, p3, ϵ}])[[1]]

Out[205]= {-0.414454, 0.959092, -0.339513, 0.165411}

          {-0.4144541824871411‘, 0.9590923197873218‘, -0.3395129038105771‘, 0.16541096501128538‘}

In[206]:= ((fi /@ xi) - (f /@ xi)) / (2 + (f /@ xi))

Out[206]= {0.000445061, -0.000445061, 0.000445061, -0.000445061, 0.000445061}

In[207]:= xi = Join[{xi[[1]]},
           # - ((D[(fi[x] - f[x]) / (2 + f[x]), x] / D[(fi[x] - f[x]) / (2 + f[x]), x, x] /. x → #) & /@ #) &[
            Take[xi, {2, -2}]], {xi[[-1]]}]

Out[207]= {-0.5, -0.392297, -0.0763497, 0.316139, 0.5}

In[208]:= Plot[(fi[x] - f[x]) / (2 + f[x]), {x, -0.5, 0.5}]

Out[208]=
```




```
In[209]:= Round[pi * {16 384, 8192, 4096, 2048}]

Out[209]= {-6790, 7857, -1391, 339}

In[210]:= Round[pi * 16 384]

Out[210]= {-6790, 15 714, -5563, 2710}

In[211]:= log2p[x_] := Module[{n},
            n = x - 32 768 - 16 384;
            Floor[(-6758 + mq15[n, 15 715 + mq15[n, -5563 + mq15[n, 2708]]]) / 64]
          ]
```

In[212]:=
```
Plot[(log2p[x] - Log[2, x / 65536] * 256) / ((2 + Log[2, x / 65536]) * 256), {x, 32768, 65535}]
```

Out[212]=



In[213]:=
```
Sqrt[Mean[Table[
   N[((log2p[x] - Log[2, x / 65536] * 256) / ((2 + Log[2, x / 65536]) * 256))^2], {x, 32768, 65535}]]]
```

Out[213]= 0.000827046

In[214]:=
```
Max[Table[
  Abs[N[(log2p[x] - Log[2, x / 65536] * 256) / ((2 + Log[2, x / 65536]) * 256)]], {x, 32768, 65535}]]
```

Out[214]= 0.00231846

In[215]:=
```
Max[Table[Abs[N[log2p[x] - Log[2, x / 65536] * 256]], {x, 32768, 65535}]]
```

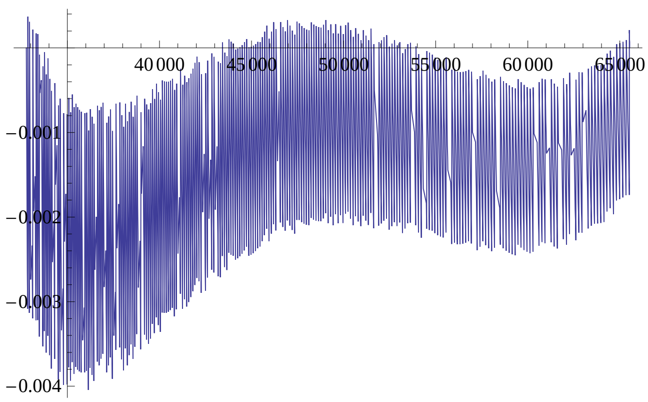
Out[215]= 0.71076

In[216]:=
```
(*This was JM's original approximation,
which was pretty good except for the lack of rounding bias at the end.*)
```

In[217]:=
```
log2p[x_] := Module[{n},
  n = x - 32768 - 16384;
  Floor[(-6791 + mq14[n, 7872 + mq14[n, -1392 + mq14[n, 319]]]) / 64]
 ]
```

In[218]:=
```
Plot[(log2p[x] - Log[2, x / 65536] * 256) / ((2 + Log[2, x / 65536]) * 256), {x, 32768, 65535}]
```

Out[218]=



In[219]:=
```
Sqrt[Mean[Table[
   N[((log2p[x] - Log[2, x / 65536] * 256) / ((2 + Log[2, x / 65536]) * 256))^2], {x, 32768, 65535}]]]
```

Out[219]= 0.00165048

In[220]:=
```
Max[Table[
  Abs[N[(log2p[x] - Log[2, x / 65 536] * 256) / ((2 + Log[2, x / 65 536]) * 256)]], {x, 32 768, 65 535}]]
```

Out[220]= 0.00407666

In[221]:=
```
Max[Table[Abs[N[log2p[x] - Log[2, x / 65 536] * 256]], {x, 32 768, 65 535}]]
```

Out[221]= 1.19962

In[222]:=
```
(*Polynomial approximation of the binary exponential.*)
```

In[223]:=
```
f[x_] := 2^x
```

In[224]:=
```
xi = N[(Table[-Cos[i * π / 4], {i, 0, 4}]) + 1] / 2
```

Out[224]= {0., 0.146447, 0.5, 0.853553, 1.}

In[237]:=
```
pi = {p0, p1, p2, p3} /.
  (Solve[({p0, p1, p2, p3}.{1, #, #^2, #^3} & /@ xi) - (f /@ xi) == {ϵ, -ϵ, ϵ, -ϵ, ϵ} * (f /@ xi),
     {p0, p1, p2, p3, ϵ}])[[1]]
```

Out[237]= {0.999925, 0.695834, 0.226067, 0.0780245}

```
{0.9999252185627103`, 0.6958335404948245`, 0.2260671554272475`, 0.07802452264063828`}
```

In[238]:=
```
((fi /@ xi) - (f /@ xi)) / (f /@ xi)
```

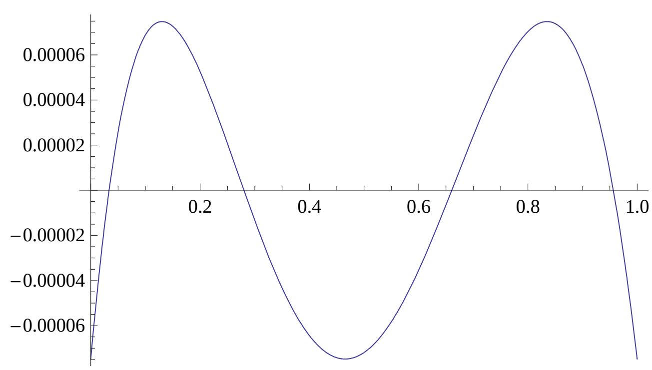
Out[238]= {-0.0000747814, 0.0000747814, -0.0000747814, 0.0000747814, -0.0000747814}

In[239]:=
```
xi =
 Join[{xi[[1]]}, # - ((D[(fi[x] - f[x]) / f[x], x] / D[(fi[x] - f[x]) / f[x], x, x] /. x → #) & /@ #) &[
   Take[xi, {2, -2}]], {xi[[-1]]}]
```

Out[239]= {0., 0.130276, 0.465445, 0.834978, 1.}

In[240]:=
```
Plot[(fi[x] - f[x]) / f[x], {x, 0, 1}]
```

Out[240]=



In[241]:=
```
Round[pi * 16 384]
```

Out[241]= {16 383, 11 401, 3704, 1278}

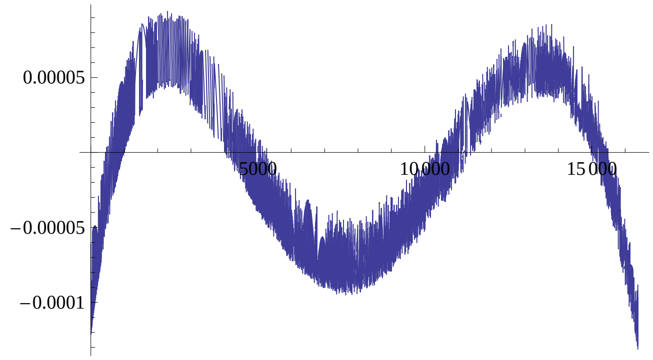
In[242]:=
```
Round[pi * {16 384, 32 768, 65 536, 65 536 * 2}]
```

Out[242]= {16 383, 22 801, 14 816, 10 227}

In[243]:=
```
exp2p[x_] :=
 16 383 + mq15[x, 22 804 + mq15[x, 14 819 + mq15[x, 10 204]]]
```

In[244]:=
```
Plot[(exp2p[x] - 2^(x / 16 384) * 16 384) / (2^(x / 16 384) * 16 384), {x, 0, 16 383}]
```

Out[244]=



In[245]:=
```
Sqrt[
 Mean[Table[N[((exp2p[x] - 2^(x / 16 384) * 16 384) / (2^(x / 16 384) * 16 384))^2], {x, 0, 16 383}]]]
```

Out[245]= 0.0000499743

In[246]:=
```
Max[Table[Abs[N[(exp2p[x] - 2^(x / 16 384) * 16 384) / (2^(x / 16 384) * 16 384)]], {x, 0, 16 383}]]
```

Out[246]= 0.000117927

In[247]:=
```
Max[Table[Abs[N[exp2p[x] - 2^(x / 16 384) * 16 384]], {x, 0, 16 384}]]
```

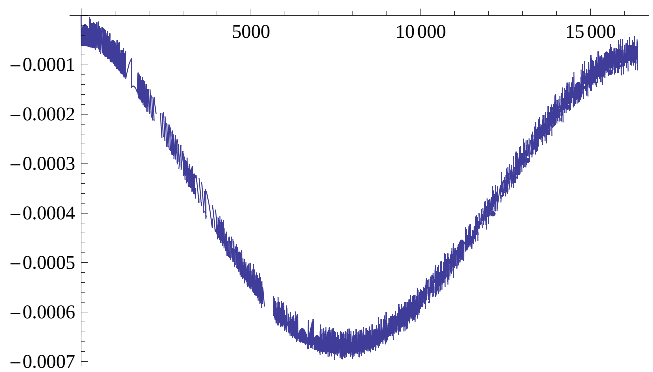
Out[247]= 3.82733

In[248]:=
```
(*This was JM’s original approximation.*)
```

In[249]:=
```
exp2p[x_] :=
 16 384 + mq14[x, 11 356 + mq14[x, 3726 + mq14[x, 1301]]]
```

In[250]:=
```
Plot[(exp2p[x] - 2^(x / 16 384) * 16 384) / (2^(x / 16 384) * 16 384), {x, 0, 16 383}]
```

Out[250]=



In[251]:=
```
Sqrt[Mean[
  Table[Abs[N[((exp2p[x] - 2^(x / 16 384) * 16 384) / (2^(x / 16 384) * 16 384))^2]], {x, 0, 16 383}]]]
```

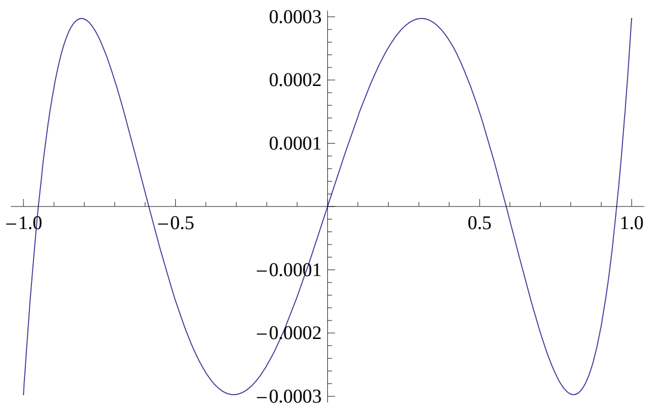
Out[251]= 0.000432746

In[252]:=
```
Max[Table[Abs[N[(exp2p[x] - 2^(x / 16 384) * 16 384) / (2^(x / 16 384) * 16 384)]], {x, 0, 16 383}]]
```

Out[252]= 0.000693476

```
In[253]:= Max[Table[Abs[N[exp2p[x] - 2^(x / 16 384) * 16 384]], {x, 0, 16 384}]]

Out[253]= 16.0832

In[254]:= (*Polynomial approximation of the reciprocal.*)

In[255]:= f[x_] := 4 / (x + 3)

In[256]:= xi = N[(Table[-Cos[i * π / 5], {i, 0, 5}])]

Out[256]= {-1., -0.809017, -0.309017, 0.309017, 0.809017, 1.}

In[263]:= pi = {p0, p1, p2, p3, p4} /.
            (Solve[({p0, p1, p2, p3, p4}.{1, #, #^2, #^3, #^4} & /@ xi) - (f /@ xi) ==
                {ϵ, -ϵ, ϵ, -ϵ, ϵ, -ϵ} * (f /@ xi), {p0, p1, p2, p3, p4, ϵ}])[[1]]

Out[263]= {1.33333, -0.442462, 0.147487, -0.0570919, 0.0190306}

In[264]:= ((fi /@ xi) - (f /@ xi)) / (f /@ xi)

Out[264]= {-0.000297354, 0.000297354, -0.000297354, 0.000297354, -0.000297354, 0.000297354}

In[265]:= xi =
           Join[{xi[[1]]}, # - ((D[(fi[x] - f[x]) / f[x], x] / D[(fi[x] - f[x]) / f[x], x, x] /. x → #) & /@ #) &[
             Take[xi, {2, -2}]], {xi[[-1]]}]

Out[265]= {-1., -0.809017, -0.309017, 0.309017, 0.809017, 1.}

In[266]:= Plot[(fi[x] - f[x]) / f[x], {x, -1, 1}]
```

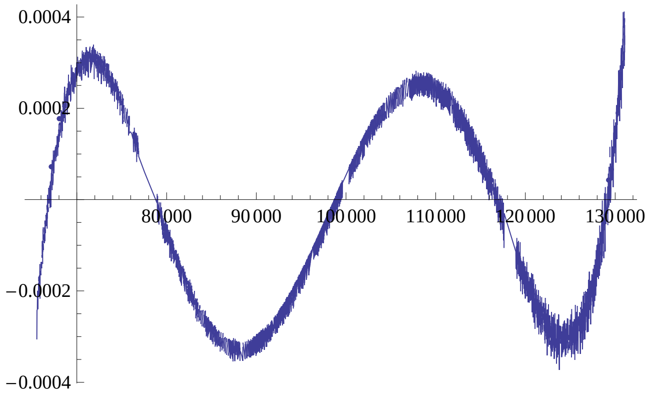
Out[266]=




```
In[267]:= Round[pi * 16 384]

Out[267]= {21 845, -7249, 2416, -935, 312}

In[268]:= rcpp[x_] := Module[{n},
            n = x - 65 536 - 32 768;
            21 845 + mq15[n, -7249 + mq15[n, 2416 + mq15[n, -935 + mq15[n, 315]]]]
           ]
```

In[269]:=
```
Plot[(rcpp[x] - (65536 / x) * 32768) / ((65536 / x) * 32768), {x, 65536, 131071}]
```

Out[269]=



In[270]:=
```
Sqrt[Mean[
  Table[Abs[N[((rcpp[x] - (65536 / x) * 32768) / ((65536 / x) * 32768))^2]], {x, 65536, 131071}]]]
```

Out[270]= 0.000213052

In[271]:=
```
Max[Table[Abs[N[(rcpp[x] - (65536 / x) * 32768) / ((65536 / x) * 32768)]], {x, 65536, 131071}]]
```

Out[271]= 0.00040389

In[272]:=
```
Max[Table[Abs[N[rcpp[x] - (65536 / x) * 32768]], {x, 65536, 131071}]]
```

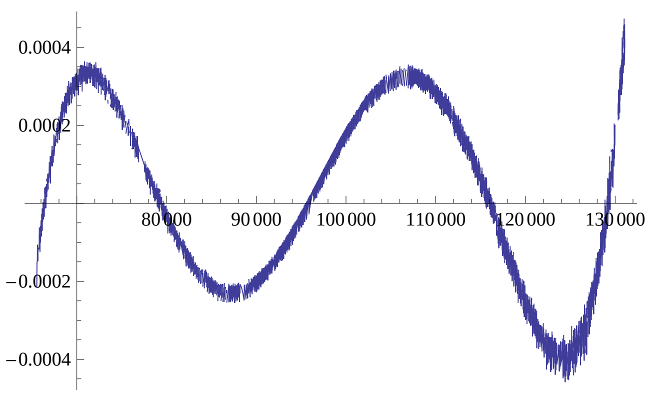
Out[272]= 10.2927

In[273]:=
```
(*This was JM's original approximation,
which was quite good. The major discrepancy is that it optimized for absolute,
not relative error.*)
```

In[274]:=
```
rcpp[x_] := Module[{n},
  n = x - 65536 - 32768;
  21848 + mq15[n, -7251 + mq15[n, 2403 + mq15[n, -934 + mq15[n, 327]]]]
 ]
```

In[275]:=
```
Plot[(rcpp[x] - (65536 / x) * 32768) / ((65536 / x) * 32768), {x, 65536, 131071}]
```

Out[275]=



In[276]:=
```
Sqrt[Mean[
  Table[Abs[N[((rcpp[x] - (65536 / x) * 32768) / ((65536 / x) * 32768))^2]], {x, 65536, 131071}]]]
```

Out[276]= 0.000228586

In[277]:=
```
Max[Table[Abs[N[(rcpp[x] - (65536 / x) * 32768) / ((65536 / x) * 32768)]], {x, 65536, 131071}]]
```

Out[277]= 0.00047091

In[278]:=
```
Max[Table[Abs[N[rcpp[x] - (65536 / x) * 32768]], {x, 65536, 131071}]]
```
Out[278]= 11.3349

In[279]:=
```
(*Reciprocal approximation using linear interpolation followed by
 two rounds of Newton's method. This gets us to the level of truncation
 error with just 1 more multiply than the polynomial approximation.*)
```

In[280]:=
```
f[x_] := 2 / (x + 1)
```

In[281]:=
```
xi = N[(Table[-Cos[i * π / 2], {i, 0, 2}]) + 1] / 2
```
Out[281]= {0., 0.5, 1.}

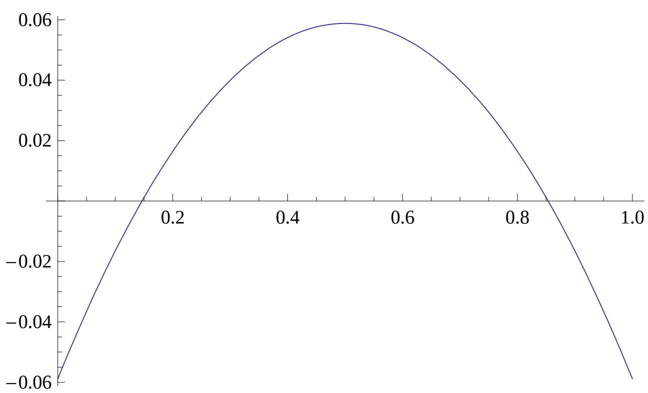
In[282]:=
```
pi =
 {p0, p1} /. (Solve[({p0, p1}.{1, #} & /@ xi) - (f /@ xi) == {ϵ, -ϵ, ϵ} * (f /@ xi), {p0, p1, ϵ}])[[1]]
```
Out[282]= {1.88235, -0.941176}

```
{1.8823529411764706`, -0.9411764705882353`}
```

In[283]:=
```
((fi /@ xi) - (f /@ xi)) / (f /@ xi)
```
Out[283]= {-0.0588235, 0.0588235, -0.0588235}

In[284]:=
```
xi =
 Join[{xi[[1]]}, # - ((D[(fi[x] - f[x]) / f[x], x] / D[(fi[x] - f[x]) / f[x], x, x] /. x → #) & /@ #) &[
   Take[xi, {2, -2}]], {xi[[-1]]}]
```
Out[284]= {0., 0.5, 1.}

In[285]:=
```
Plot[(fi[x] - f[x]) / f[x], {x, 0, 1}]
```
Out[285]=

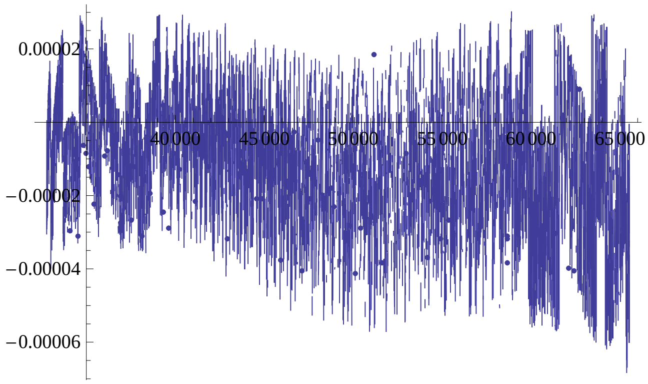


In[286]:=
```
Round[pi * 16384]
```
Out[286]= {30840, -15420}

In[287]:=
```
rcp[x_] := Module[{n, r},
  n = x - 32768;
  r = 30840 + mq15[-15420, n];
  r = r - mq15[r, mq15[n, r] + (r - 32768)];
  r = r - (mq15[r, mq15[n, r] + (r - 32768)] + 1)
 ]
```

```
In[288]:= Plot[(rcp[x] - (32768 / x) * 32768) / ((32768 / x) * 32768), {x, 32768, 65535}]
```

Out[288]=




```
In[289]:= Sqrt[Mean[
            Table[Abs[N[((rcp[x] - (32768 / x) * 32768) / ((32768 / x) * 32768))^2]], {x, 32768, 65535}]]]

Out[289]= 0.0000214418

In[290]:= Max[Table[Abs[N[(rcp[x] - (32768 / x) * 32768) / ((32768 / x) * 32768)]], {x, 32768, 65535}]]

Out[290]= 0.0000705346

In[291]:= Max[Table[Abs[N[rcp[x] - (32768 / x) * 32768]], {x, 32768, 65535}]]

Out[291]= 1.24665
```